Chapter 19

Heart of Algebra

Heart of Algebra focuses on the mastery of linear equations, systems of linear equations, and linear functions. The ability to analyze and create linear equations, inequalities, and functions is essential for success in college and careers, as is the ability to solve linear equations and systems fluently. The questions in Heart of Algebra include both multiple-choice questions and student-produced response questions. On some questions, the use of a calculator is not permitted; on other questions, the use of a calculator is allowed.

The questions in Heart of Algebra vary significantly in form and appearance. They may be straightforward fluency exercises or pose challenges of strategy or understanding, such as interpreting the relationship between graphical and algebraic representations or solving as a process of reasoning. You will be required to demonstrate both procedural skill and a deep understanding of concepts.

Let's explore the content and skills assessed by Heart of Algebra questions.

REMEMBER

The SAT Math Test will require you to demonstrate a deep understanding of several core algebra topics, namely linear equations, systems of linear equations, and linear functions. These topics are fundamental to the learning and work often required in college and career.

Linear Equations, Linear Inequalities, and Linear Functions in Context

When you use algebra to analyze and solve a problem in real life, a key step is to represent the context of the problem algebraically. To do this, you may need to define one or more variables that represent quantities in the context. Then you need to write one or more expressions, equations, inequalities, or functions that represent the relationships described in the context. For example, once you write an equation that represents the context, you solve the equation. Then you interpret the solution to the equation in terms of the context. Questions on the SAT Math Test may assess your ability to accomplish any or all of these steps.

EXAMPLE 1

In 2014, County X had 783 miles of paved roads. Starting in 2015, the county has been building 8 miles of new paved roads each year. At this rate, how many miles of paved road will County X have in 2030? (Assume that no paved roads go out of service.)

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Many Heart of Algebra questions such as this one will require you to accomplish the following steps:

1. Define one or more variables that represent quantities in the question.
2. Write one or more equations, expressions, inequalities, or functions that represent the relationships described in the question.
3. Solve the equation, and interpret the solution in terms of what the question is asking.

Ample practice with each of these steps will help you develop your math skills and knowledge.

The first step in answering this question is to decide what variable or variables you need to define. The question is asking how the number of miles of paved road in County X depends on the year. This can be represented using n, the number of years after 2014. Then, since the question says that County X had 783 miles of paved road in 2014 and is building 8 miles of new paved roads each year, the expression $783 + 8n$ gives the number of miles of paved roads in County X in year n. The year 2030 is $2030 - 2014 = 16$ years after 2014; thus, the year 2030 corresponds to $n = 16$. Hence, to find the number of miles of paved roads in County X in 2030, substitute 16 for n in the expression $783 + 8n$, giving $783 + 8(16) = 783 + 128 = 911$. Therefore, at the given rate of building, County X will have 911 miles of paved roads in 2030.

There are different questions that can be asked about the same context.

EXAMPLE 2

In 2014, County X had 783 miles of paved roads. Starting in 2015, the county has been building 8 miles of new paved roads each year. At this rate, if n is the number of years after 2014, which of the following functions f gives the number of miles of paved road there will be in County X? (Assume that no paved roads go out of service.)

A) $f(n) = 8 + 783n$
B) $f(n) = 2{,}014 + 783n$
C) $f(n) = 783 + 8n$
D) $f(n) = 2{,}014 + 8n$

REMEMBER

There are several different ways you can be tested on the same underlying algebra concepts. Practicing a variety of questions, with different contexts, is a good way to ensure you'll be ready for the questions you'll come across on the SAT.

This question already defines the variable and asks you to create a function that describes the context. The discussion in Example 1 shows that the correct answer is choice C.

EXAMPLE 3

In 2014, County X had 783 miles of paved roads. Starting in 2015, the county has been building 8 miles of new paved roads each year. At this rate, in which year will County X first have at least 1,000 miles of paved roads? (Assume that no paved roads go out of service.)

In this question, you must solve an inequality. As in Example 1, let n be the number of years after 2014. Then the expression $783 + 8n$ gives the number of miles of paved roads in County X. The question is asking when there will first be at least 1,000 miles of paved roads in County X. This condition can be represented by the inequality $783 + 8n \geq 1{,}000$. To find the year in which there will first be at least 1,000 miles of paved roads, you solve this inequality for n. Subtracting 783 from each side of $783 + 8n \geq 1{,}000$ gives $8n \geq 217$. Then dividing each side of $8n \geq 217$ gives $n \geq 27.125$. Note that an important part of relating the inequality $783 + 8n \geq 1{,}000$ back to the context is to notice that n is counting calendar years and so it must be an integer. The least value of n that satisfies $783 + 8n \geq 1{,}000$ is 27.125, but the year $2014 + 27.125 = 2041.125$ does not make sense as an answer, and in 2041, there would be only $783 + 8(27) = 999$ miles of paved roads in the county. Therefore, the variable n needs to be rounded up to the next integer, and so the least possible value of n is 28. Therefore, the year that County X will first have at least 1,000 miles of paved roads is 28 years after 2014, or 2042.

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Solving an equation or inequality is often only part of the problem-solving process. You must also interpret the solution in the context of the question, so be sure to remind yourself of the question's context and the meaning of the variables you solved for before selecting your answer.

In Example 1, once the variable n was defined, you needed to find an expression that represents the number of miles of paved road in terms of n. In other questions, creating the correct expression, equation, or function may require a more insightful understanding of the context.

EXAMPLE 4

To edit a manuscript, Miguel charges \$50 for the first 2 hours and \$20 per hour after the first 2 hours. Which of the following expresses the amount in dollars, C, Miguel charges if it takes him x hours to edit a manuscript, where $x > 2$?

A) $C = 20x$
B) $C = 20x + 10$
C) $C = 20x + 50$
D) $C = 20x + 90$

The question defines the variables C and x and asks you to express C in terms of x. To create the correct expression, you must note that since the \$50 that Miguel charges pays for his first 2 hours of editing, he charges \$20 per hour only *after* the first 2 hours. Thus, if it takes x hours for Miguel to edit a manuscript, he charges \$50 for the first 2 hours and \$20 per hour for the remaining time, which is $x - 2$ hours. Thus, his total charge, C, can be written as $C = 50 + 20(x - 2)$. This does not match any of the choices. But when the right-hand side of $C = 50 + 20(x - 2)$ is expanded, you get $C = 50 + 20x - 40$, or $C = 20x + 10$, which is choice B.

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When the solution you arrive at doesn't match any of the answer choices, consider if expanding, simplifying, or rearranging your solution will cause it to match an answer choice. Often, this extra step is needed to arrive at the correct answer.

As with Examples 1 to 3, there are different questions that could be asked about this context. For example, you could be asked to find how long it took Miguel to edit a manuscript if he charged \$370.

Absolute Value

Absolute value expressions, inequalities, and equations are included in Heart of Algebra. (Graphs of absolute value equations and functions are in Passport to Advanced Math.) One definition of absolute value is

$$|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$

The absolute value of any real number is nonnegative. An important consequence of this definition is that $|-x| = |x|$ for any real number x. Another important consequence of this definition is that if a and b are any two real numbers, then $|a - b|$ is equal to the distance between a and b on the number line.

EXAMPLE 5

REMEMBER

The SAT Math Test will require you to have a deep understanding of absolute value, not simply that the absolute value of any real number is nonnegative. Example 5, for instance, requires you to apply your knowledge of two concepts, absolute value and inequality, to represent the relationship described in a specific context.

The stratosphere is the layer of the Earth's atmosphere that is more than 10 kilometers (km) and less than 50 km above the Earth's surface. Which of the following inequalities describes all possible heights x, in km, above the Earth's surface that are in the stratosphere?

A) $|x + 10| < 50$
B) $|x - 10| < 50$
C) $|x + 30| < 20$
D) $|x - 30| < 20$

The question states that the stratosphere is the layer of the Earth's atmosphere that is greater than 10 km and less than 50 km above the Earth's surface. Thus, the possible heights x, in km, above the Earth's surface that are in the stratosphere are given by the inequality $10 < x < 50$. To answer the question, you need to find an absolute value inequality that is equivalent to $10 < x < 50$.

The inequality $10 < x < 50$ describes the open interval (10, 50). To describe an interval with an absolute value inequality, use the midpoint and the size of the interval. The midpoint of (10, 50) is $\frac{10 + 50}{2} = 30$. Then observe that the interval (10, 50) consists of all points that are within 20 of the midpoint. That is, (10, 50) consists of x, whose distance from 30 on the number line is less than 20. The distance between x and 30 on the number line is $|x - 30|$. Therefore, the possible values of x are described by $|x - 30| < 20$, which is choice D.

Systems of Linear Equations and Inequalities in Context

You may need to define more than one variable and create more than one equation or inequality to represent a context and answer a question. There are questions on the SAT Math Test that require you to create and solve a system of equations or create a system of inequalities.

EXAMPLE 6

> Maizah bought a pair of pants and a briefcase at a department store. The sum of the prices before sales tax was \$130.00. There was no sales tax on the pants and a 9% sales tax on the briefcase. The total Maizah paid, including the sales tax, was \$136.75. What was the price, in dollars, of the pants?

To answer the question, you first need to define the variables. The question discusses the prices of a pair of pants and a briefcase and asks you to find the price of the pants. So it is appropriate to let P be the price of the pants, in dollars, and to let B be the price of the briefcase, in dollars. Since the sum of the prices before sales tax was \$130.00, the equation $P + B = 130$ is true. A sales tax of 9% was added to the price of the briefcase. Since 9% is equal to 0.09, the price of the briefcase with tax was $B + 0.09B = 1.09B$. There was no sales tax on the pants, and the total Maizah paid, including tax, was \$136.75, so the equation $P + 1.09B = 136.75$ holds.

Now, you need to solve the system

$$P + B = 130$$
$$P + 1.09B = 136.75$$

Subtracting the sides of the first equation from the corresponding sides of the second equation gives you $(P + 1.09B) - (P + B) = 136.75 - 130$, which simplifies to $0.09B = 6.75$. Now you can divide each side of $0.09B = 6.75$ by 0.09. This gives you $B = \frac{6.75}{0.09} = 75$. This is the value of B, the price, in dollars, of the briefcase. The question asks for the price, in dollars, of the pants, which is P. You can substitute 75 for B in the equation $P + B = 130$, which gives you $P + 75 = 130$, or $P = 130 - 75 = 55$, so the pants cost \$55.

(Note that this example has no choices. It is a student-produced response question. On the SAT, you would grid your answer in the spaces provided on the answer sheet.)

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You can use either of two approaches — combination or substitution — when solving a system of linear equations. One may get you to the answer more quickly than the other, depending on the equations you're working with and what you're solving for. Practice using both to give you greater flexibility on test day.

REMEMBER

While this question may seem complex, as it involves numerous steps, solving it requires a strong understanding of the same underlying principles outlined above: defining variables, creating equations to represent relationships, solving equations, and interpreting the solution.

EXAMPLE 7

Each morning, John jogs at 6 miles per hour and rides a bike at 12 miles per hour. His goal is to jog and ride his bike a total of at least 9 miles in less than 1 hour. If John jogs j miles and rides his bike b miles, which of the following systems of inequalities represents John's goal?

A) $\frac{j}{6} + \frac{b}{12} < 1$
$j + b \geq 9$

B) $\frac{j}{6} + \frac{b}{12} \geq 1$
$j + b < 9$

C) $6j + 12b \geq 9$
$j + b < 1$

D) $6j + 12b < 1$
$j + b \geq 9$

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In Example 7, the answer choices each contain two parts. Use this to your advantage by tackling one part at a time and eliminating answers that don't work.

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You should be able to quickly rearrange three-part equations such as the rate equation (rate = distance / time) for any of the three parts. Example 7 requires you to solve the equation for time.

John jogs j miles and rides his bike b miles; his goal to jog and ride his bike a total of at least 9 miles is represented by the inequality $j + b \geq 9$. This eliminates choices B and C.

Since rate × time = distance, it follows that time is equal to distance divided by rate. John jogs j miles at 6 miles per hour, so the time he jogs is equal to $\frac{j \text{ miles}}{6 \text{ miles/hour}} = \frac{j}{6}$ hours. Similarly, since John rides his bike b miles at 12 miles per hour, the time he rides his bike is $\frac{b}{12}$ hours. Thus, John's goal to complete his jog and his bike ride in less than 1 hour can be represented by the inequality $\frac{j}{6} + \frac{b}{12} = < 1$. The system $j + b \geq 9$ and $\frac{j}{6} + \frac{b}{12} < 1$ is choice A.

Fluency in Solving Linear Equations, Linear Inequalities, and Systems of Linear Equations

Creating linear equations, linear inequalities, and systems of linear equations that represent a context is a key skill for success in college and careers. It is also essential to be able to fluently solve linear equations, linear inequalities, and systems of linear equations. Some of the questions in the Heart of Algebra section of the SAT Math Test present equations, inequalities, or systems without a context and directly assess your fluency in solving them.

Some fluency questions allow the use of a calculator; other questions do not permit the use of a calculator and test your ability to solve equations, inequalities, and systems of equations by hand. Even for questions where a calculator is allowed, you may be able to answer the question more quickly without using a calculator, such as in Example 9. Part of what the SAT Math Test assesses is your ability to decide when using a calculator to answer a question is appropriate. Example 8 is an example of a question that could appear on either the calculator or no-calculator portion of the Math Test.

EXAMPLE 8

$$3\left(\frac{1}{2} - y\right) = \frac{3}{5} + 15y$$

What is the solution to the equation above?

Expanding the left-hand side of the equation gives $\frac{3}{2} - 3y = \frac{3}{5} + 15y$, which can be rewritten as $18y = \frac{3}{2} - \frac{3}{5}$. Multiplying each side of $18y = \frac{3}{2} - \frac{3}{5}$ by 10, the least common multiple of 2 and 5, clears the denominators: $180y = \frac{30}{2} - \frac{30}{5} = 15 - 6 = 9$. Therefore, $y = \frac{9}{180} = \frac{1}{20}$.

EXAMPLE 9

$$-2(3x - 2.4) = -3(3x - 2.4)$$

What is the solution to the equation above?

You could solve this in the same way as Example 8, by multiplying everything out and simplifying. But the structure of the equation reveals that -2 times a quantity, $3x - 2.4$, is equal to -3 times the same quantity. This is only possible if the quantity $3x - 2.4$ is equal to zero. Thus, $3x - 2.4 = 0$, or $3x = 2.4$. Therefore, the solution is $x = 0.8$.

EXAMPLE 10

$$-2x = 4y + 6$$

$$2(2y + 3) = 3x - 5$$

What is the solution (x, y) to the system of equations above?

This is an example of a system you can solve quickly by substitution. Since $-2x = 4y + 6$, it follows that $-x = 2y + 3$. Now you can substitute $-x$ for $2y + 3$ in the second equation. This gives you $2(-x) = 3x - 5$, which simplifies to $5x = 5$, or $x = 1$. Substituting 1 for x in the first equation gives you $-2 = 4y + 6$, which simplifies to $4y = -8$, or $y = -2$. Therefore, the solution to the system is $(1, -2)$.

REMEMBER

While a calculator is permitted on one portion of the SAT Math Test, it's important to not over-rely on a calculator. Some questions, such as Example 9, can be solved more efficiently without using a calculator. Your ability to choose when to use and when not to use a calculator is one of the things the SAT Math Test assesses, so be sure to practice this in your studies.

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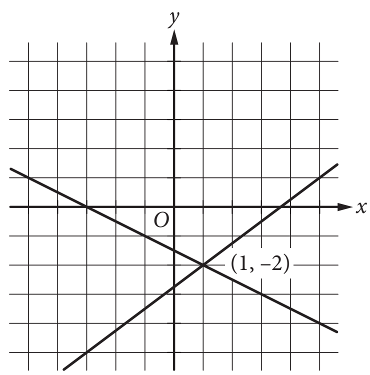
In Example 6, the combination approach yields an efficient solution to the question. In Example 10, substitution turns out to be a fast approach. These examples illustrate the benefits of knowing both approaches and thinking critically about which approach may be faster on a given question.

In the preceding examples, you have found a unique solution to linear equations and to systems of two linear equations in two variables. But not all such equations and systems have solutions, and some have infinitely many solutions. Some questions on the SAT Math Test assess your ability to determine whether an equation or a system has one solution, no solutions, or infinitely many solutions.

The Relationships among Linear Equations, Lines in the Coordinate Plane, and the Contexts They Describe

A system of two linear equations in two variables can be solved by graphing the lines in the coordinate plane. For example, you can graph the system of equations in Example 10 in the *xy*-plane:



The point of intersection gives the solution to the system.

If the equations in a system of two linear equations in two variables are graphed, each graph will be a line. There are three possibilities:

1. The lines intersect in one point. In this case, the system has a unique solution.
2. The lines are parallel. In this case, the system has no solution.
3. The lines are identical. In this case, every point on the line is a solution, and so the system has infinitely many solutions.

By putting the equations in the system into slope-intercept form, the second and third cases can be identified. If the lines have the same slope and different y-intercepts, they are parallel; if both the slope and the y-intercept are the same, the lines are identical.

How are the second and third cases represented algebraically? Examples 11 and 12 concern this question.

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Graphing systems of two linear equations is another effective approach to solving them. Practice arranging linear equations into $y = mx + b$ form and graphing them in the coordinate plane.

EXAMPLE 11

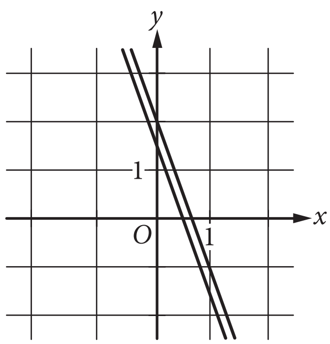
$$2y + 6x = 3$$
$$y + 3x = 2$$

How many solutions (x, y) are there to the system of equations above?

A) Zero
B) One
C) Two
D) More than two

If you multiply each side of $y + 3x = 2$ by 2, you get $2y + 6x = 4$. Then subtracting each side of $2y + 6x = 3$ from the corresponding side of $2y + 6x = 4$ gives $0 = 1$. This is a false statement. Therefore, the system has zero solutions (x, y).

Alternatively, you could graph the two equations. The graphs are parallel lines, so there are no points of intersection.



REMEMBER

When the graphs of a system of two linear equations are parallel lines, as in Example 11, the system has zero solutions. If the question states that a system of two linear equations has an infinite number of solutions, as in Example 12, the equations must be equivalent.

EXAMPLE 12

$$3s - 2t = a$$
$$-15s + bt = -7$$

In the system of equations above, a and b are constants. If the system has infinitely many solutions, what is the value of a?

If a system of two linear equations in two variables has infinitely many solutions, the two equations in the system must be equivalent. Since the two equations are presented in the same form, the second equation must be equal to the first equation multiplied by a constant. Since the coefficient of s in the second equation is -5 times the coefficient of s in the first equation, multiply each side of the first equation by -5. This gives you the system

$$-15s + 10t = -5a$$
$$-15s + bt = -7$$

Since these two equations are equivalent and have the same coefficient of s, the coefficients of t and the constants on the right-hand side must also be the same. Thus, $b = 10$ and $-5a = -7$. Therefore, the value of a is $\frac{7}{5}$.

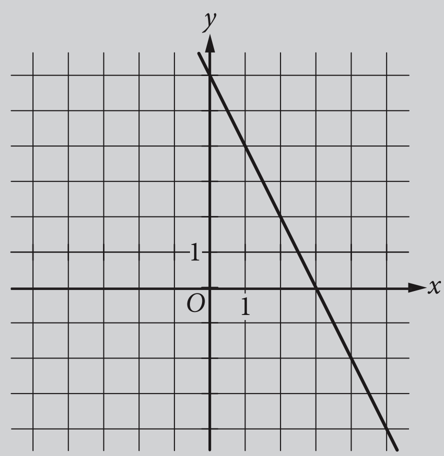
There will also be questions on the SAT Math Test that assess your knowledge of the relationship between the algebraic and the geometric representations of a line, that is, between an equation of a line and its graph. The key concepts are:

- If the slopes of line ℓ and line k are each defined (that is, if neither line is a vertical line), then
  - Line ℓ and line k are parallel if and only if they have the same slope.
  - Line ℓ and line k are perpendicular if and only if the product of their slopes is -1.

REMEMBER

The SAT Math Test will further assess your understanding of linear equations by, for instance, asking you to select a linear equation that describes a given graph, select a graph that describes a given linear equation, or determine how a graph may be impacted by a change in its equation.

EXAMPLE 13



The graph of line k is shown in the xy-plane above. Which of the following is an equation of a line that is perpendicular to line k?

A) $y = -2x + 1$

B) $y = -\frac{1}{2}x + 2$

C) $y = \frac{1}{2}x + 3$

D) $y = 2x + 4$

Note that the graph of line k passes through the points (0, 6) and (3, 0). Thus, the slope of line k is $\frac{0-6}{3-0} = -2$. Since the product of the slopes of perpendicular lines is −1, a line that is perpendicular to line k will have slope $\frac{1}{2}$. All the choices are in slope-intercept form, and so the coefficient of x is the slope of the line represented by the equation. Therefore, choice C, $y = \frac{1}{2}x + 3$, is an equation of a line with slope $\frac{1}{2}$, and thus this line is perpendicular to line k.

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Example 13 requires a strong understanding of slope as well as the ability to calculate slope: slope = rise / run = change in x / change in y. Parallel lines have slopes that are equal. Perpendicular lines have slopes whose product is −1.

As we've noted, some contexts can be described with a linear equation. The graph of a linear equation is a line. A line has geometric properties such as its slope and its y-intercept. These geometric properties can often be interpreted in terms of the context. The SAT Math Test has questions that assess your ability to make these interpretations. For example, look back at the contexts in Examples 1 to 3. You created a linear function, $f(n) = 783 + 8n$, that describes the number of miles of paved road County X will have n years after 2014. This equation can be graphed in the coordinate plane, with n on the horizontal axis and $f(n)$ on the vertical axis. This graph is a line with slope 8 and vertical intercept 783. The slope, 8, gives the number of miles of new paved roads added each year, and the vertical intercept gives the number of miles of paved roads in 2014, the year that corresponds to $n = 0$.

EXAMPLE 14

A voter registration drive was held in Town Y. The number of voters, V, registered T days after the drive began can be estimated by the equation $V = 3{,}450 + 65T$. What is the best interpretation of the number 65 in this equation?

A) The number of registered voters at the beginning of the registration drive
B) The number of registered voters at the end of the registration drive
C) The total number of voters registered during the drive
D) The number of voters registered each day during the drive

The correct answer is choice D. For each day that passes, it is the next day of the registration drive, and so T increases by 1. When T increases by 1, the value of $V = 3{,}450 + 65T$ increases by 65. That is, the number of voters registered increased by 65 for each day of the drive. Therefore, 65 is the number of voters registered each day during the drive.

You should note that choice A describes the number 3,450, and the numbers described by choices B and C can be found only if you know how many days the registration drive lasted; this information is not given in the question.

Mastery of linear equations, systems of linear equations, and linear functions is built upon key skills such as analyzing rates and ratios. Several key skills are discussed in the next domain, Problem Solving and Data Analysis.

Chapter 20

Problem Solving and Data Analysis

The Problem Solving and Data Analysis section of the SAT Math Test assesses your ability to use your math understanding and skills to solve problems set in the real world. Problem Solving and Data Analysis questions test your ability to create a representation of a problem, consider the units involved, pay attention to the meaning of quantities, know and use different properties of mathematical operations and representations, and apply key principles of statistics. Special focus in this domain will be given to mathematical models. You may be asked to create and use a model and to understand the distinction between the model predictions and data collected. Models are a representation of real life. They help us to explain or interpret the behavior of certain components of a system and to predict future results that are as yet unobserved or unmeasured.

The questions involve quantitative reasoning about ratios, rates, and proportional relationships and may require understanding and applying unit rates. Many of the problems are set in academic and career settings and draw from science, including the social sciences.

Some questions present information about the relationship between two variables in a graph, scatterplot, table, or another form and ask you to analyze and draw conclusions about the given information. The questions assess your understanding of the key properties of, and the differences between, linear, quadratic, and exponential relationships and how these properties apply to the corresponding real-life contexts. An important example is understanding the difference between simple interest and compound interest.

Problem Solving and Data Analysis also includes questions that assess your understanding of essential concepts in statistics. You may be asked to analyze univariate data presented in bar graphs, histograms, line graphs, and box-and-whisker plots, or bivariate data presented in scatterplots and two-way tables. This includes computing and interpreting measures of center, interpreting measures of spread, describing overall patterns, and recognizing the effects of outliers on measures of center. These questions may test your understanding of the conceptual meaning of standard deviation (although you will not be asked to calculate a standard deviation).

Other questions may ask you to estimate the probability of a simple or compound event, employing different approaches, rules, or probability models. Special attention is given to the notion of conditional probability, which is tested using two-way tables or other contexts.

Some questions require the ability to draw conclusions about an entire population from a random sample of that population and how variability affects those conclusions. The questions may test your understanding of randomization-based inference and the conceptual meaning of the margin of error (although you will not be asked to calculate a margin of error) when the mean or the proportion of a population is estimated using sample data. You may be presented with a description of a study and asked to explain what types of conclusions can be drawn with regard to relationships between variables involved and to what population can the study findings be appropriately generalized.

REMEMBER

Problem Solving and Data Analysis comprise 17 of the 58 questions (29%) on the Math Test.

The questions in Problem Solving and Data Analysis include both multiple-choice questions and student-produced response questions. The use of a calculator is allowed for all questions in this domain.

Problem Solving and Data Analysis is one of the three SAT Math Test subscores, reported on a scale of 1 to 15.

Let's explore the content and skills assessed by Problem Solving and Data Analysis questions.

Ratio, Proportion, Units, and Percentage

Ratio and proportion is one of the major ideas in mathematics. Introduced well before high school, ratio and proportion is a theme throughout mathematics, in applications, in careers, in college mathematics courses, and beyond.

EXAMPLE 1

On Thursday, 240 adults and children attended a show. The ratio of adults to children was 5 to 1. How many children attended the show?

A) 40
B) 48
C) 192
D) 200

Because the ratio of adults to children was 5 to 1, there were 5 adults for every 1 child. In fractions, $\frac{5}{6}$ of the 240 who attended were adults and $\frac{1}{6}$ were children. Therefore, $\frac{1}{6} \times 240 = 40$ children attended the show, which is choice A.

Ratios on the SAT may be expressed in the form 3 to 1, 3:1, $\frac{3}{1}$, or simply 3.

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A ratio represents the proportional relationship between quantities, not the actual quantities themselves. Fractions are an especially effective way to represent and work with ratios.

EXAMPLE 2

On an architect's drawing of the floor plan for a house, 1 inch represents 3 feet. If a room is represented on the floor plan by a rectangle that has sides of lengths 3.5 inches and 5 inches, what is the actual floor area of the room in square feet?

A) 17.5
B) 51.0
C) 52.5
D) 157.5

Because 1 inch represents 3 feet, the actual dimensions of the room are $3 \times 3.5 = 10.5$ feet and $3 \times 5 = 15$ feet. Therefore, the floor area of the room is $10.5 \times 15 = 157.5$ square feet, which is choice D.

Another classic example of ratio is the length of a shadow. At a given location and time of day, it might be true that a fence post that is 4 feet high casts a shadow that is 6 feet long. This ratio of the height of the object to the length of the shadow, 4 to 6 or $\frac{2}{3}$, remains the same for any object at the same location and time. So, for example, a person who is 6 feet tall would cast a shadow that is $\frac{3}{2} \times 6 = 9$ feet long. In this situation, in which one variable quantity is always a fixed constant times another variable quantity, the two quantities are said to be directly proportional.

Variables x and y are said to be directly proportional if $y = kx$, where k is a nonzero constant. The constant k is called the constant of proportionality.

In the preceding example, you would say the length of an object's shadow is directly proportional to the height of the object, with constant of proportionality $\frac{3}{2}$. So if you let L be the length of the shadow and H be the height of the object, then $L = \frac{3}{2}H$.

Notice that both L and H are lengths, so the constant of proportion, $\frac{L}{H} = \frac{3}{2}$, has no units. In contrast, let's consider Example 2 again. On the scale drawing, 1 inch represents 3 feet. The length of an actual measurement is directly proportional to its length on the scale drawing. But to find the constant of proportionality, you need to keep track of units: $\frac{3 \text{ feet}}{1 \text{ inch}} = \frac{36 \text{ inches}}{1 \text{ inch}} = 36$.

Hence, if S is a length on the scale drawing that corresponds to an actual length of A, then $A = 36S$.

Many of the questions on the SAT Math Test require you to pay attention to units. Some questions in Problem Solving and Data Analysis require you to convert units either between the English system and the metric system or within those systems.

EXAMPLE 3

Scientists estimate that the Pacific Plate, one of Earth's tectonic plates, has moved about 1,060 kilometers in the past 10.3 million years. What was the average speed of the Pacific Plate during that time period, in centimeters per year?

A) 1.03
B) 10.3
C) 103
D) 1,030

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Pay close attention to units, and convert units if required by the question. Writing out the unit conversion as a series of multiplication steps, as seen here, will help ensure accuracy. Intermediate units should cancel (as do the kilometers and meters in Example 3), leaving you with the desired unit (centimeters per year).

Since 1 kilometer = 1,000 meters and 1 meter = 100 centimeters, you get $\frac{1{,}060 \text{ kilometers}}{10{,}300{,}000 \text{ years}} \times \frac{1{,}000 \text{ meters}}{1 \text{ kilometer}} \times \frac{100 \text{ centimeters}}{1 \text{ meter}} = 10.3$ centimeters per year.

Therefore, the correct answer is choice B.

Questions may require you to move between unit rates and total amounts.

EXAMPLE 4

County Y consists of two districts. One district has an area of 30 square miles and a population density of 370 people per square mile, and the other district has an area of 50 square miles and a population density of 290 people per square mile. What is the population density, in people per square mile, for all of County Y?

REMEMBER

13 of the 58 questions on the Math Test, or 22%, are student-produced response questions in which you will grid your answers in the spaces provided on the answer sheet.

(Note that this example has no choices. It is a student-produced response question. On an SAT, you would grid your answer in the spaces provided on the answer sheet.)

The first district has an area of 30 square miles and a population density of 370 people per square mile, so its total population is $30 \text{ square miles} \times 370 \frac{\text{people}}{\text{square mile}} = 11{,}100$ people. The other district has an area of 50 square miles and a population density of 290 people per square mile, so its total population is $50 \text{ square miles} \times 290 \frac{\text{people}}{\text{square mile}} = 14{,}500$ people.

Thus, County Y has total population 11,100 + 14,500 = 25,600 people and total area 30 + 50 = 80 square miles. Therefore, the population density of County Y is $\frac{25,600}{80}$ = 320 people per square mile.

Problem Solving and Data Analysis also includes questions involving percentages, which are a type of proportion. These questions may involve the concepts of percentage increase and percentage decrease.

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Percent is a type of proportion that means "per 100." 20%, for instance, means 20 out of (or per) 100. Percent increase or decrease is calculated by finding the difference between two quantities, then dividing the difference by the original quantity and multiplying by 100.

EXAMPLE 5

A furniture store buys its furniture from a wholesaler. For a particular table, the store usually charges its cost from the wholesaler plus 75%. During a sale, the store charged the wholesale cost plus 15%. If the sale price of the table was $299, what is the usual price for the table?

A) $359
B) $455
C) $479
D) $524

The sale price of the table was $299. This is equal to the cost from the wholesaler plus 15%. Thus, $299 = 1.15(wholesale cost), and the cost from the wholesaler is $\frac{\$299}{1.15}$ = $260. Therefore, the usual price the store charges for the table is 1.75 × $260 = $455, which is choice B.

Interpreting Relationships Presented in Scatterplots, Graphs, Tables, and Equations

The behavior of a variable and the relationship between two variables in a real-world context may be explored by considering data presented in tables and graphs.

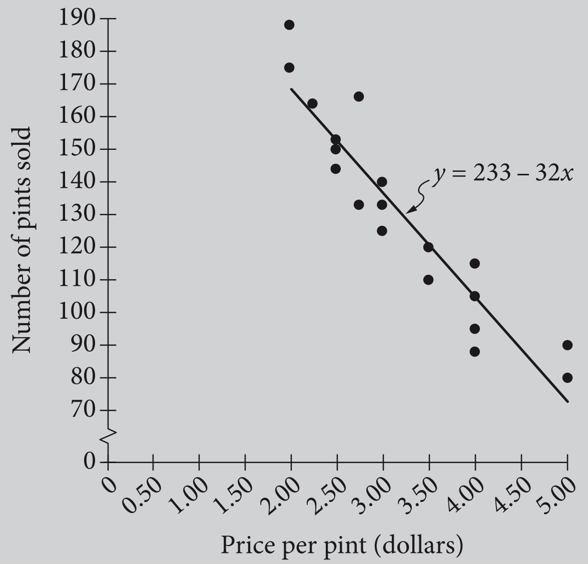
The relationship between two variables may be modeled by a function or equation. The function or equation may be found by examining ordered pairs of data values and by analyzing how the variables are related to one another in the real world. The model may allow very accurate predictions, as for example models used in physical sciences, or may only describe a trend, with considerable variability between the actual and predicted values, as for example models used in behavioral and social sciences.

REMEMBER

The ability to interpret and synthesize data from charts, graphs, and tables is a widely applicable skill in college and in many careers and thus is tested on the SAT Math Test.

Questions on the SAT Math Test assess your ability to understand and analyze the relationships between two variables, the properties of the functions used to model these relationships, and the conditions under which a model is considered to be good, acceptable, or inappropriate. The questions in Problem Solving and Data Analysis focus on linear, quadratic, and exponential relationships.

EXAMPLE 6



A grocery store sells pints of raspberries and sets the price per pint each week. The scatterplot above shows the price and the number of pints of raspberries sold for 19 weeks, along with the line of best fit and the equation for the line of best fit.

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A line of best fit is a straight line that best represents the data on a scatterplot. It is written in $y = mx + b$ form.

There are several different questions that could be asked about this context.

A. According to the line of best fit, how many pints of raspberries would the grocery store expect to sell in a week when the price of raspberries is \$4.50 per pint?

Because the line of best fit has equation $y = 233 - 32x$, where x is the price, in dollars, for a pint of raspberries and y is the number of pints of raspberries sold, the number of pints the store would be expected to sell in a week where the price of raspberries is \$4.50 per pint is $233 - 32(4.50) = 89$ pints.

B. For how many of the 19 weeks shown was the number of pints of raspberries sold greater than the amount predicted by the line of best fit?

For a given week, the number of pints of raspberries sold is greater than the amount predicted by the line of best fit if and only if the point representing that week lies above the line of best fit. Of the 19 points, 8 lie above the line of best fit, so there were 8 weeks in which the number of pints sold was greater than what was predicted by the line of best fit.

C. What is the best interpretation of the meaning of the slope of the line of best fit?

On the SAT, this question would be followed by multiple-choice answer options. The slope of the line of best fit is -32. This means that the correct answer would state that for each dollar that the price of a pint of raspberries increases, the store expects to sell 32 fewer pints of raspberries.

D. What is the best interpretation of the meaning of the y-intercept of the line of best fit?

On the SAT, this question would be followed by multiple-choice answer options.

In this context, the y-intercept does not represent a likely scenario, so it cannot be accurately interpreted in terms of this context. According to the model, the y-intercept means that if the store sold raspberries for $0 per pint — that is, if the store gave raspberries away — 173 people would be expected to accept the free raspberries. However, it is not realistic that the store would give away raspberries, and if they did, it is likely that far more people would accept the free raspberries.

The fact that the y-intercept indicates that 173 people would accept free raspberries is one limitation of the model. Another limitation is that for a price of $7.50 per pint or above, the model predicts that a negative number of people would buy raspberries, which is impossible. In general, you should be cautious about applying a model for values outside of the given data. In this example, you should only be confident in the prediction of sales for prices between $2 and $5.

Giving a line of best fit, as in this example, assumes that the relationship between the variables is best modeled by a linear function, but that is not always true. On the SAT, you may see data that are best modeled by a linear, quadratic, or exponential model.

(**Note**: Questions interpreting the slope and intercepts of a line of best fit, such as in **C** and **D**, may be classified as part of the Heart of Algebra section and contribute to the Heart of Algebra subscore.)

EXAMPLE 7

Time (hours)	Number of bacteria
0	1×10^3
1	4×10^3
2	1.6×10^4
3	6.4×10^4

The table above gives the initial number (at time $t = 0$) of bacteria placed in a growth medium and the number of bacteria in the growth medium over 3 hours. Which of the following functions models the number of bacteria, $N(t)$, after t hours?

A) $N(t) = 4{,}000t$
B) $N(t) = 1{,}000 + 3{,}000t$
C) $N(t) = 1{,}000(4^{-t})$
D) $N(t) = 1{,}000(4^{t})$

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To determine if a model is linear or exponential, examine the change in the quantity between successive time periods. If the difference in quantity is constant, the model is linear. If the ratio in the quantity is constant (for instance, 4 times greater than the preceding time period), then the model is exponential.

The given choices are linear and exponential models. If a quantity is increasing linearly with time, then the *difference* in the quantity between successive time periods is constant. If a quantity is increasing exponentially with time, then the *ratio* in the quantity between successive time periods is constant. According to the table, after each hour, the number of bacteria in the culture is 4 times as great as it was the preceding hour: $\frac{4 \times 10^3}{1 \times 10^3} = \frac{1.6 \times 10^4}{4 \times 10^3} = \frac{6.4 \times 10^4}{1.6 \times 10^4} = 4$. That is, for each increase of 1 in t, the value of $N(t)$ is multiplied by 4. At $t = 0$, which corresponds to the time when the culture was placed in the medium, there were 103 bacteria. This is modeled by the exponential function $N(t) = 1{,}000(4^t)$, which has value 1,000 at $t = 0$ and increases by a factor of 4 for each increase of 1 in the value of t. Choice D is the correct answer.

The SAT Math Test may have questions on simple and compound interest, which are important examples of linear and exponential growth, respectively.

EXAMPLE 8

A bank has opened a new branch and, as part of a promotion, the bank branch is offering \$1,000 certificates of deposit at simple interest of 4% per year. The bank is selling certificates with terms of 1, 2, 3, or 4 years. Which of the following functions gives the total amount, A, in dollars, a customer will receive when a certificate with a term of k years is finally paid?

A) $A = 1{,}000(1.04k)$
B) $A = 1{,}000(1 + 0.04k)$
C) $A = 1{,}000(1.04)^k$
D) $A = 1{,}000(1 + 0.04^k)$

For 4% simple interest, 4% of the original deposit is added to the original deposit for each year the deposit was held. That is, if the certificate has a term of k years, $4k$% is added to the original deposit to get the final amount. Because $4k$% is $0.04k$, the final amount paid to the customer is $A = 1{,}000 + 1{,}000(0.04k) = 1{,}000(1 + 0.04k)$. Choice B is the correct answer.

The general formula for simple interest is $A = P(1 + rt)$, where P is the original deposit, called the principal; r is the annual interest rate expressed as a decimal; and t is the length the deposit is held. In Example 8, $P = \$1{,}000$, $r = 0.04$, and $t = k$ years; so A, in dollars, is given by $A = 1{,}000[1 + (0.04)k]$.

In contrast, compound interest is an example of exponential growth.

EXAMPLE 9

A bank has opened a new branch and, as part of a promotion, the bank branch is offering \$1,000 certificates of deposit at an interest rate of 4% per year, compounded semiannually. The bank is selling certificates with terms of 1, 2, 3, or 4 years. Which of the following functions gives the total amount, A, in dollars, a customer will receive when a certificate with a term of k years is finally paid?

A) $A = 1{,}000(1 + 0.04k)$
B) $A = 1{,}000(1 + 0.08k)$
C) $A = 1{,}000(1.04)^k$
D) $A = 1{,}000(1.02)^{2k}$

The interest is compounded semiannually, that is, twice a year. At the end of the first half year, 2% of the original deposit is added to the value of the certificate (4% annual interest multiplied by the time period, which is $\frac{1}{2}$ year, gives 2% interest). When the interest is added, the value, in dollars, of the certificate is now $1{,}000 + 1{,}000(0.02) = 1{,}000(1.02)$. Since the interest is reinvested (compounded), the new principal at the beginning of the second half year is $1{,}000(1.02)$. At the end of the second half year, 2% of $1{,}000(1.02)$ is added to the value of the certificate; the value, in dollars, of the certificate is now $1{,}000(1.02) + 1{,}000(1.02)(0.02)$, which is equal to $1{,}000(1.02)(1.02) = 1{,}000(1.02)^2$. In general, after n compounding periods, the amount, A, in dollars, is $A = 1{,}000(1.02)^n$.

When the certificate is paid after k years, the value of the certificate will have been multiplied by the factor (1.02) a total of $2k$ times. Therefore, the total amount, A, in dollars, a customer will receive when a certificate with a term of k years is finally paid is $A = 1{,}000(1.02^{2k})$. Choice D is the correct answer.

The general formula for compound interest is $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where P is the principal, r is the annual interest rate expressed as a decimal, t is the number of years the deposit is held, and n is the number of times the interest is compounded per year. In Example 9, $P = \$1{,}000$, $r = 0.04$, $t = k$, and $n = 2$; so A, in dollars, is given by $A = 1{,}000\left(1 + \frac{0.04}{2}\right)^{2k} = 1{,}000(1.02)^{2k}$.

Note: Although the stated interest rate is 4% per year in Example 9, the value of the account increases by more than 4% in a year, namely 4.04% per year. (You may have seen banks offer an account in this way, for example, 5.00% annual interest rate, 5.13% effective annual yield.) If you take calculus, you will often see a situation in which a stated rate of change differs from the change over an interval. But on the SAT, other than compound interest,

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Know the formulas for simple and compound interest.

Simple interest: $A = P(1 + rt)$

Compound interest: $A = P(1 + r/n)^{nt}$

A is the total amount, P is the principal, r is the interest rate expressed as a decimal, t is the time period, and n is the number of times the interest is compounded per year.

the stated rate of change is always equal to the actual rate of change. For example, if a question says that the height of a plant increases by 10% each month, it means that $\frac{\text{height of the plant now}}{\text{height of the plant a month ago}} = 1.1$ (or if a question says that the population of a city is decreasing by 3% per year, it means that $\frac{\text{population of the city now}}{\text{population of the city a year ago}} = 0.97$. Then, if the question asks by what percentage the height of the plant will increase in 2 months, you can write

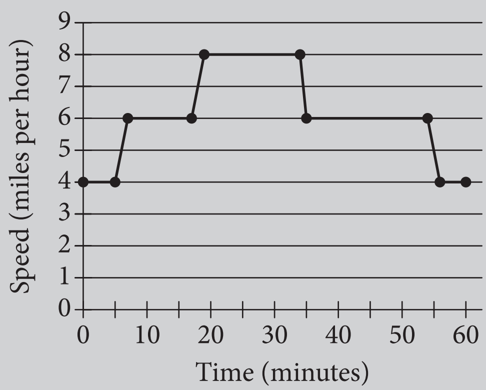
$$\frac{\text{height of the plant in 2 months}}{\text{height of the plant now}} = \frac{\text{height of the plant in 2 months}}{\text{height of the plant in 1 month}} + \frac{\text{height of the plant in 1 month}}{\text{height of the plant now}}$$

$$= 1.1 \times 1.1 = 1.21$$

Therefore, the answer is that the height of the plant increases by 21% in 2 months.

An SAT Math Test question may ask you to interpret a graph that shows the relationship between two variables.

EXAMPLE 10



Each evening, Maria walks, jogs, and runs for a total of 60 minutes. The graph above shows Maria's speed during the 60 minutes. Which segment of the graph represents the times when Maria's speed is the greatest?

A) The segment from (17, 6) to (19, 8)
B) The segment from (19, 8) to (34, 8)
C) The segment from (34, 8) to (35, 6)
D) The segment from (35, 6) to (54, 6)

The correct answer is choice B. Because the vertical coordinate represents Maria's speed, the part of the graph with the greatest vertical coordinate represents the times when Maria's speed is the greatest. This is the highest part of the graph, the segment from (19, 8) to (34, 8), when Maria runs at 8 miles per hour (mph). Choice A represents the time during which Maria's speed is increasing from 6 to 8 mph; choice C represents the time during which Maria's speed is decreasing from 8 to 6 mph; and choice D represents the longest period of Maria moving at the same speed, not the times when Maria's speed is the greatest.

More Data and Statistics

Some questions on the SAT Math Test will assess your ability to understand and analyze data presented in a table, bar graph, histogram, line graph, or other display.

EXAMPLE 11

A store is deciding whether to install a new security system to prevent shoplifting. The security manager of the store estimates that 10,000 customers enter the store each week, 24 of whom will attempt to shoplift. The manager estimates the results of the new security system in detecting shoplifters would be as shown in the table below.

	Alarm sounds	Alarm does not sound	Total
Customer attempts to shoplift	21	3	24
Customer does not attempt to shoplift	35	9,941	9,976
Total	56	9,944	10,000

According to the manager's estimates, if the alarm sounds for a customer, what is the probability that the customer did *not* attempt to shoplift?

A) 0.03%
B) 0.35%
C) 0.56%
D) 62.5%

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Probability is the measure of how likely an event is. When calculating the probability of an event, use the following formula:

probability = number of favorable (or desired) outcomes / total number of possible outcomes

According to the manager's estimates, the alarm will sound for 56 customers. Of these 56 customers, 35 did *not* attempt to shoplift. Therefore, if the alarm sounds, the probability that the customer did *not* attempt to shoplift is $\frac{35}{56} = \frac{5}{8} = 62.5\%$. The correct answer is choice D.

Example 11 is an example of a conditional probability.

You may be asked to answer questions that involve a measure of center for a data set: the average (arithmetic mean) or the median. A question may ask you to draw conclusions about one or more of these measures of center even if the exact values cannot be calculated. To recall briefly:

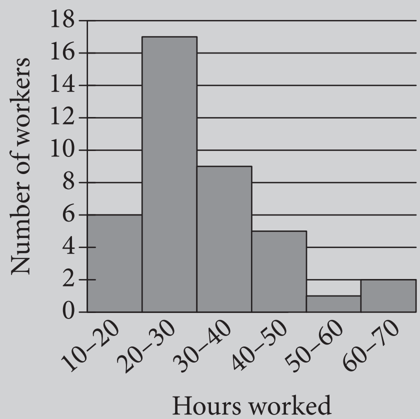
The mean of a set of numerical values is the sum of all the values divided by the number of values in the set.

REMEMBER

Mean, median, and mode are measures of center for a data set, while range and standard deviation are measures of spread.

The median of a set of numerical values is the middle value when the values are listed in increasing (or decreasing) order. If the set has an even number of values, then the median is the average of the two middle values.

EXAMPLE 12



The histogram above summarizes the number of hours worked last week by the 40 employees of a landscaping company. In the histogram, the first bar represents all workers who worked at least 10 hours but less than 20 hours; the second represents all workers who worked at least 20 hours but less than 30 hours; and so on. Which of the following could be the median and mean number of hours worked for the 40 employees?

A) Median = 22, Mean = 23
B) Median = 24, Mean = 22
C) Median = 26, Mean = 32
D) Median = 32, Mean = 30

(**Note**: On the SAT, all histograms have the same type of boundary condition. That is, the values represented by a bar include the left endpoint but do not include the right endpoint.)

If the number of hours the 40 employees worked is listed in increasing order, the median will be the average of the 20th and the 21st numbers on the list. The first 6 numbers on the list will be workers represented by the first bar; hence, each of the first 6 numbers will be at least 10 but less than 20. The next 17 numbers, that is, the 7th through the 23rd numbers on the list, will be workers represented by the second bar; hence, each of the next 17 numbers will be at least 20 but less than 30. Thus, the 20th and the 21st numbers on the list will be at least 20 but less than 30. Therefore, any of the median values in choices A, B, or C are possible, but the median value in choice D is not.

Now let's find the possible values of the mean. Each of the 6 employees represented by the first bar worked at least 10 hours but less than 20 hours. Thus, the total number of hours worked by these 6 employees is at least 60. Similarly, the total number of hours worked by the 17 employees represented by the second bar is at least 340; the total number of hours worked by the 9 employees represented by the third bar is at least 270; the total number of hours worked by the 5 employees represented by the fourth bar is at least 200; the total number of hours worked by the 1 employee represented by the fifth bar is at least 50; and the total number of hours worked by the 2 employees represented by the sixth bar is at least 120. Adding all these hours up shows that the total number of hours worked by all 40 employees is at least $60 + 340 + 270 + 200 + 50 + 120 = 1{,}040$. Therefore, the mean number of hours worked by all 40 employees is at least $\frac{1{,}040}{40} = 26$. Therefore, only the values of the average given in choices C and D are possible. Because only choice C has possible values for both the median and the mean, it is the correct answer.

A data set may have a few values that are much larger or smaller than the rest of the values in the set. These values are called *outliers*. An outlier may represent an important piece of data. For example, if a data set consists of rates of a certain illness in various cities, a data point with a very high value could indicate a serious health issue to be investigated.

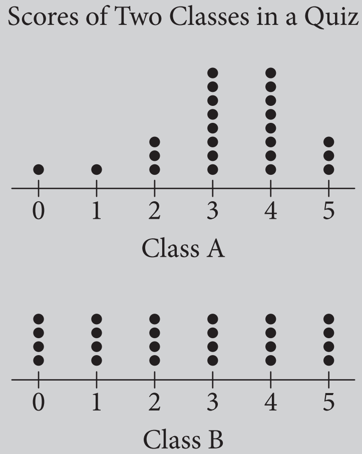
In general, outliers affect the mean but not the median. Therefore, outliers that are larger than the rest of the points in the data set tend to make the mean greater than the median, and outliers that are smaller than the rest of the points in the data set tend to make the mean less than the median. The most evident graphical display used to identify outliers is the box plot.

The mean and the median are different ways to describe the center of a data set. Another key characteristic of a data set is the amount of variation, or spread, in the data. One measure of spread is the *standard deviation*, which is a measure of how far away the points in the data set are from the average value. On the SAT Math Test, you will *not* be asked to compute the standard deviation of a data set, but you do need to understand that a larger standard deviation corresponds to a data set whose values are more spread out from the mean value.

REMEMBER

You will not be asked to calculate the exact standard deviation of a set of data on the SAT Math Test, but you will be expected to demonstrate an understanding of what standard deviation measures.

EXAMPLE 13



The dot plots above summarize the scores that two classes, each with 24 students, at Central High School achieved on a current events quiz. Which of the following correctly compares the standard deviation of the scores in each of the classes?

A) The standard deviation of the scores in Class A is smaller.
B) The standard deviation of the scores in Class B is smaller.
C) The standard deviation of the scores in Class A and Class B is the same.
D) The relationship cannot be determined from the information given.

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When asked to compare the standard deviations of two data sets, first approximate the mean of each data set. Then, ask yourself which data set has values that are more closely clustered around the mean. That data set will have the smaller standard deviation.

In Class A, the large majority of scores are 3 and 4, with only a few scores of 0, 1, 2, and 5; the average score is between 3 and 4. In Class B, the scores are evenly spread out across all possible scores, with many scores not close to the average score, which is 2.5. Because the scores in Class A are more closely clustered around the mean, the standard deviation of the scores in Class A is smaller. The correct answer is choice A.

A *population parameter* is a numerical value that describes a characteristic of a population. For example, the percentage of registered voters who would vote for a certain candidate is a parameter describing the population of registered voters in an election. Or the average income of a household for a city is a parameter describing the population of households in that city. An essential purpose of statistics is to estimate a population parameter based on a sample from the population. A common example is election polling, where researchers will interview a random sample of registered voters in an election to estimate the outcome of an election. The precision of the estimate depends on the variability of the data and the sample size. For example, if household incomes in a city vary widely or the sample is small, the estimate that comes from a sample may differ considerably from the actual value for the population (the parameter).

For example, suppose you want to estimate the average amount of time each week that the students at a high school spend on the Internet. Suppose the high school has 1,200 students. It would be time consuming to ask all 1,200 students, but you can ask a sample of students. Suppose you have time to ask 80 students. Which 80 students? In order to have a sample that is representative of the population, students who will participate in the study should be selected at random. That is, each student must have the same chance to be selected. Randomization is essential in protecting against bias and helps to calculate the sampling error reliably. This can be done in different ways. You could write each student's name on a slip of paper, put all the slips in a bowl, mix up the slips, and then draw 80 names from the bowl. In practice, a computer is often used to select a random sample.

If you do not select a random sample, it may introduce bias. For example, if you found 80 students from those attending a game of the school's football team, those people would be more likely to be interested in sports, and in turn, an interest in sports might affect the average amount of time the students spend on the Internet. The result would be that the average time those 80 students spend on the Internet might not be an accurate estimate of the average amount of time *all* students at the school spend on the Internet.

Suppose you select 80 students at random from the 1,200 students at the high school. You ask them how much time they spend on the Internet each week, and you find that the average time is 14 hours. You also find that 6 of the 80 students spend less than 2 hours each week on the Internet. How can these results be used to make a generalization about the entire population of 1,200 students?

Because the sample was selected at random, the average of 14 hours is the most reasonable estimate for average time on the Internet for all 1,200 students. Also, you can use proportional reasoning to estimate the number of students at the school who spend less than 2 hours on the Internet each week. Because 6 of the 80 students in the sample spend less than 2 hours per week on the Internet, the best estimate for the number of students in the entire school who spend less than 2 hours is x students, where $\frac{x}{1{,}200} = \frac{6}{80}$. Solving this equation for x gives $x = 90$. So it is appropriate to estimate that 90 of the 1,200 students at the school spend less than 2 hours per week on the Internet.

But this is not all. An essential part of statistics is accounting for the variability of the estimate. The estimates above are reasonable, but they are unlikely to be exactly correct. Statistical analysis can also describe how far from the estimates the actual values are likely to be. To describe the precision of an estimate, statisticians use *margins of error* and *confidence intervals*. On the SAT, you will not be expected to compute a margin of error or a confidence interval, but you should understand how different factors affect the margin of error and how to interpret a given margin of error or confidence interval in the context.

REMEMBER

You will not need to calculate margins of error or confidence intervals on the SAT Math Test, but you should understand what these concepts mean and be able to interpret them in context.

If the example above were an SAT question, you might be told that the estimate of an average of 14 hours per week on the Internet from the random sample of 80 students has margin of error 1.2 hours at 95% confidence level.

This means that in random samples of size 80, the actual average will be within 1.2 hours of the true average in 95% of possible samples. In terms of confidence intervals, you can be 95% confident that the interval from 12.8 hours to 15.2 hours includes the true average amount of time on the Internet for all students at the school. (**Note:** In statistics, confidence levels other than 95% can be used, but SAT questions will always use 95% confidence levels.)

There are some key points to note.

1. When the confidence level is kept the same, the size of the margin of error is affected by two factors: the variability in the data and the sample size. The larger the standard deviation, the larger the margin of error; the smaller the standard deviation, the smaller the margin of error. Increasing the size of the random sample provides more information and reduces the margin of error.
2. The margin of error and the confidence interval apply to the estimated value of the parameter for the entire population, *not* for the value of the variable for particular individuals. In the example, we are 95% confident that the interval from 12.8 to 15.2 hours includes *the true average* amount of time on the Internet for all students at the school. It does not imply that 95% of students spend between 12.8 and 15.2 hours on the Internet.

EXAMPLE 14

A quality control researcher at an electronics company is testing the life of the company's batteries in a certain camera. The researcher selects 100 batteries at random from the daily output of the batteries and finds that the average life of the batteries has a 95% confidence interval of 324 to 360 camera pictures. Which of the following conclusions is the most reasonable based on the confidence interval?

A) 95% of all the batteries produced by the company that day have a life between 324 and 360 pictures.
B) 95% of all the batteries ever produced by the company have a life between 324 and 360 pictures.
C) It is plausible that the true average life of batteries produced by the company that day is between 324 and 360 pictures.
D) It is plausible that the true average life of all the batteries ever produced by the company is between 324 and 360 pictures.

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When a confidence interval is provided, determine the value for which the interval applies. Confidence intervals concern the average value of a population and do not apply to values of individual objects in the population.

The correct answer is choice C. Choices A and B are incorrect because the confidence interval gives information about the true *average* life of all batteries produced by the company that day, not about the life of any individual battery. Choice D is incorrect because the sample of batteries was taken from the population of all of the batteries produced by the company on that day. The population of all batteries the company ever produced may have a different average life because of changes in the formulation of the batteries, wear on machinery, improvements in production processes, and many other factors.

The statistics examples discussed so far are largely based on investigations intended to estimate some characteristic of a population: the amount of time students spend on the Internet, the life of a battery, and the percentage of registered voters who plan to vote for a candidate. Another primary focus of statistics is to investigate relationships between variables and to draw conclusions about cause and effect. For example, does a new type of physical therapy help people recover from knee surgery faster? For such a study, some people who have had knee surgery will be randomly assigned to the new therapy, while other people who have had knee surgery will be randomly assigned to the usual therapy. The medical results of these patients can be compared. The key questions from a statistical viewpoint are:

- Can the results appropriately be generalized from the sample of patients in the study to the entire population of people who are recovering from knee surgery?
- Do the results allow one to appropriately conclude that the new therapy *caused* any difference in the results for the two groups of patients?

The answers depend on the use of random sampling and random assignment of individuals into groups of different conditions.

- If the sample of all subjects in a study were selected at random from the entire population in question, the results can appropriately be generalized to the entire population because random sampling ensures that each individual has the same chance to be selected for the sample.
- If the subjects in the sample were randomly assigned to treatments, it may be appropriate to make conclusions about cause and effect because the treatment groups will be roughly equivalent at the beginning of the experiment other than the treatment they receive.

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In order for results of a study to be generalized to the entire population, and for a cause-and-effect relationship to be established, both random sampling and random assignment of individuals to treatments is needed.

This can be summarized in the following table.

	Subjects Selected at Random	**Subjects Not Selected at Random**
Subjects randomly assigned to treatments	Results can be appropriately generalized to the entire population. Conclusions about cause and effect can appropriately be drawn.	Results *cannot* be appropriately generalized to the entire population. Conclusions about cause and effect can appropriately be drawn.
Subjects not randomly assigned to treatments	Results can be appropriately generalized to the entire population. Conclusions about cause and effect *cannot* appropriately be drawn.	Results *cannot* be appropriately generalized to the entire population. Conclusions about cause and effect *cannot* appropriately be drawn.

The previous example discussed treatments in a medical experiment. The word *treatment* refers to any factor that is deliberately varied in an experiment.

EXAMPLE 15

A community center offers a Spanish course. This year, all students in the course were offered additional audio lessons they could take at home. The students who took these additional audio lessons did better in the course than students who didn't take the additional audio lessons. Which of the following is an appropriate conclusion?

A) Taking additional audio lessons will cause an improvement for any student who takes any foreign language course.
B) Taking additional audio lessons will cause an improvement for any student who takes a Spanish course.
C) Taking additional audio lessons was the cause of the improvement for the students at the community center who took the Spanish course.
D) No conclusion about cause and effect can be made regarding students at the community center who took the additional audio lessons at home and their performance in the Spanish course.

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Be wary of conclusions that claim a cause-and-effect relationship or that generalize a conclusion to a broader population. Before accepting a conclusion, assess whether or not the subjects were selected at random from the broader population and whether or not subjects were randomly assigned treatments.

The correct answer is choice D. The better results of these students may have been a result of being more motivated, as shown in their willingness to do extra work, and not the additional audio lessons. Choice A is incorrect because no conclusion about cause and effect is possible without random assignment to treatments and because the sample was only students taking a Spanish course, so no conclusion can be appropriately made about students taking all foreign language courses. Choice B is incorrect because no conclusion about cause and effect is possible without random assignment to treatments and because the students taking a Spanish course at the community center is not a random sample of all students who take a Spanish course. Choice C is incorrect because the students taking the Spanish course at the community center were not randomly assigned to use the additional audio lessons or not use the additional audio lessons.

Chapter 21

Passport to Advanced Math

Passport to Advanced Math questions include topics that are especially important for students to master *before* studying advanced math. Chief among these topics is the understanding of the structure of expressions and the ability to analyze, manipulate, and rewrite these expressions. This section also includes reasoning with more complex equations and interpreting and building functions. Passport to Advanced Math is one of the three subscores in the SAT Math Test that are reported on a scale of 1 to 15. Questions in this section may be part of the Science subscore or part of the History and Social Studies subscore.

REMEMBER

16 of the 58 questions (28%) on the SAT Math Test are Passport to Advanced Math questions.

As you saw in Chapter 19, the questions in Heart of Algebra focus on the mastery of linear equations, systems of linear equations, and linear functions. In contrast, the questions in Passport to Advanced Math focus on the ability to work with and analyze more complex equations. The questions may require you to demonstrate procedural skill in adding, subtracting, and multiplying polynomials and in dividing a polynomial by a linear expression. You may be required to work with expressions involving exponentials, integer and rational powers, radicals, or fractions with a variable in the denominator. The questions may ask you to solve a quadratic equation, a radical equation, a rational equation, or a system consisting of a linear equation and a nonlinear equation. You may be required to manipulate an equation in several variables to isolate a quantity of interest.

Some questions in Passport to Advanced Math will ask you to build a quadratic or exponential function or an equation that describes a context or to interpret the function or solution to the equation in terms of the context.

Throughout the section, your ability to recognize structure is assessed. Expressions and equations that appear complex may use repeated terms or repeated expressions. By noticing these patterns, the complexity of a problem can be quickly simplified. Structure may be used to factor or otherwise rewrite an expression, to solve a quadratic or other equation, or to draw conclusions about the context represented by an expression, equation, or function. You may be asked to identify or derive the form of an expression or function that reveals information about the expression or function or the context it represents.

Passport to Advanced Math questions also assess your understanding of functions and their graphs. A question may require you to demonstrate your understanding of function notation, including interpreting an expression where the argument of a function is an expression rather than a variable. The questions may assess your knowledge of the domain and range of a function and your understanding of how the algebraic properties of a function relate to the geometric characteristics of its graph.

The questions in this section include both multiple-choice questions and student-produced response questions. On some questions, the use of a calculator is not permitted; on other questions, the use of a calculator is allowed. On questions where the use of a calculator is permitted, you must decide whether using your calculator is an effective strategy.

Let's consider the content and skills assessed by Passport to Advanced Math questions.

Operations with Polynomials and Rewriting Expressions

Questions on the SAT Math Test may assess your ability to add, subtract, and multiply polynomials.

EXAMPLE 1

$$(x^2 + bx - 2)(x + 3) = x^3 + 6x^2 + 7x - 6$$

In the equation above, b is a constant. If the equation is true for all values of x, what is the value of b?

A) 2
B) 3
C) 7
D) 9

REMEMBER

Passport to Advanced Math questions build on the knowledge and skills tested on Heart of Algebra questions. Develop proficiency with Heart of Algebra questions before tackling Passport to Advanced Math questions.

To find the value of b, expand the left-hand side of the equation and then collect like terms so that the left-hand side is in the same form as the right-hand side.

$$\begin{aligned}(x^2 + bx - 2)(x + 3) &= (x^3 + bx^2 - 2x) + (3x^2 + 3bx - 6) \\ &= x^3 + (3 + b)x^2 + (3b - 2)x - 6\end{aligned}$$

Since the two polynomials are equal for all values of x, the coefficient of matching powers of x should be the same. Therefore, $x^3 + (3 + b)x^2 + (3b - 2)x - 6$ and $x^3 + 6x^2 + 7x - 6$ reveals that $3 + b = 6$ and $3b - 2 = 7$. Solving either of these equations gives $b = 3$, which is choice B.

Questions may also ask you to use structure to rewrite expressions. The expression may be of a particular type, such as a difference of squares, or it may require insightful analysis.

EXAMPLE 2

Which of the following is equivalent to $16s^4 - 4t^2$?

A) $4(s^2 - t)(4s^2 + t)$
B) $4(4s^2 - t)(s^2 + t)$
C) $4(2s^2 - t)(2s^2 + t)$
D) $(8s^2 - 2t)(8s^2 + 2t)$

This example appears complex at first, but it is very similar to the equation $x^2 - y^2$ and this factors as $(x - y)(x + y)$. The expression $16s^4 - 4t^2$ is also the difference of two squares: $16s^4 - 4t^2 = (4s^2)^2 - (2t)^2$. Therefore, it can be factored as $(4s^2)^2 - (2t)^2 = (4s^2 - 2t)(4s^2 + 2t)$. This expression can be rewritten as $(4s^2 - 2t)(4s^2 + 2t) = 2(2s^2 - t)(2)(2s^2 + t) = 4(2s^2 - t)(2s^2 + t)$, which is choice C.

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Passport to Advanced Math questions require a high comfort level working with quadratic equations and expressions, including foiling and factoring. Recognizing classic quadratics such as $x^2 - y^2 = (x - y)(x + y)$ can also improve your speed and accuracy.

EXAMPLE 3

$$y^5 - 2y^4 - cxy + 6x$$

In the polynomial above, c is a constant. If the polynomial is divisible by $y - 2$, what is the value of c?

If the expression is divisible by $y - 2$, then the expression $y - 2$ can be factored from the larger expression. Since $y^5 - 2y^4 = (y - 2)y^4$, you have $y^5 - 2y^4 - cxy + 6x = (y - 2)(y^4) - cxy + 6x$. If this entire expression is divisible by $y - 2$, then $-cxy + 6x$ must be divisible by $y - 2$. Thus, $-cxy + 6x = (y - 2)(-cx) = -cxy + 2cx$. Therefore, $2c = 6$, and the value of c is 3.

Quadratic Functions and Equations

Questions in Passport to Advanced Math may require you to build a quadratic function or an equation to represent a context.

EXAMPLE 4

A car is traveling at x feet per second. The driver sees a red light ahead, and after 1.5 seconds reaction time, the driver applies the brake. After the brake is applied, the car takes $\frac{x}{24}$ seconds to stop, during which time the average speed of the car is $\frac{x}{2}$ feet per second. If the car travels 165 feet from the time the driver saw the red light to the time it comes to a complete stop, which of the following equations can be used to find the value of x?

A) $x^2 + 48x - 3{,}960$

B) $x^2 + 48x - 7{,}920$

C) $x^2 + 72x - 3{,}960$

D) $x^2 + 72x - 7{,}920$

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Example 4 requires careful translation of a word problem into an algebraic equation. It pays to be deliberate and methodical when translating word problems into equations on the SAT.

During the 1.5-second reaction time, the car is still traveling at x feet per second, so it travels a total of $1.5x$ feet. The average speed of the car during the $\frac{x}{24}$-second braking interval is $\frac{x}{2}$ feet per second, so over this interval, the car travels $\left(\frac{x}{2}\right)\left(\frac{x}{24}\right) = \frac{x^2}{48}$ feet. Since the total distance the car travels from the time the driver saw the red light to the time it comes to a complete stop is 165 feet, you have the equation $\frac{x^2}{48} + 1.5x = 165$. This quadratic equation can be rewritten in standard form by subtracting 165 from each side and then multiplying each side by 48, giving $x^2 + 72x - 7{,}920$, which is choice D.

REMEMBER

The SAT Math Test may ask you to solve a quadratic equation. Be prepared to use the appropriate method. Practice using the various methods (below) until you are comfortable with all of them.

1. Factoring
2. Completing the square
3. Quadratic formula
4. Using a calculator (if permitted)

Some questions on the SAT Math Test will ask you to solve a quadratic equation. You must determine the appropriate procedure: factoring, completing the square, the quadratic formula, use of a calculator (if permitted), or use of structure. You should also know the following facts in addition to the formulas in the directions:

- The sum of the solutions of $x^2 + bx + c = 0$ is $-b$.
- The product of the solutions of $x^2 + bx + c = 0$ is c.

Each of the facts can be seen from the factored form of a quadratic. If r and s are the solutions of $x^2 + bx + c = 0$, then $x^2 + bx + c = (x - r)(x - s)$. Thus, $b = -(r + s)$ and $c = (-r)(-s)$.

EXAMPLE 5

What are the solutions x of $x^2 - 3 = x$?

A) $\frac{-1 \pm \sqrt{11}}{2}$

B) $\frac{-1 \pm \sqrt{13}}{2}$

C) $\frac{1 \pm \sqrt{11}}{2}$

D) $\frac{1 \pm \sqrt{13}}{2}$

The equation can be solved by using the quadratic formula or by completing the square. Let's use the quadratic formula. First, subtract x from each side of $x^2 - 3 = x$ to put it in standard form: $x^2 - x - 3 = 0$. The quadratic formula states the solutions x of the equation $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. For the equation $x^2 - x - 3 = 0$, you have $a = 1$, $b = -1$, and $c = -3$. Substituting these formulas into the quadratic formula gives $x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)} = \frac{1 \pm \sqrt{1 - (-12)}}{2} = \frac{1 \pm \sqrt{13}}{2}$, which is choice D.

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The quadratic formula states that the solutions x of the equation $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

EXAMPLE 6

If $x > 0$ and $2x^2 + 3x - 2 = 0$, what is the value of x?

The left-hand side of the equation can be factored: $2x^2 + 3x - 2 = (2x - 1)(x + 2) = 0$. Therefore, either $2x - 1 = 0$, which gives $x = \frac{1}{2}$, or $x + 2 = 0$, which gives $x = -2$. Since $x > 0$, the value of x is $\frac{1}{2}$.

REMEMBER

Pay close attention to all of the details in the question. In Example 6, x can equal $\frac{1}{2}$ or -2, but since the question states that $x > 0$, the value of x must be $\frac{1}{2}$.

EXAMPLE 7

What is the sum of the solutions of $(2x - 1)^2 = (x + 2)^2$?

If a and b are real numbers and $a^2 = b^2$, then either $a = b$ or $a = -b$. Since $(2x - 1)^2 = (x + 2)^2$, either $2x - 1 = x + 2$ or $2x - 1 = -(x + 2)$. In the first case, $x = 3$, and in the second case, $3x = -1$, or $x = -\frac{1}{3}$. Therefore, the sum of the solutions x of $(2x - 1)^2 = (x + 2)^2$ is $3 + \left(-\frac{1}{3}\right) = \left(\frac{8}{3}\right)$.

Exponential Functions, Equations, and Expressions and Radicals

We examined exponential functions in Examples 7 and 8 of Chapter 20. Some questions in Passport to Advanced Math ask you to build a function that models a given context. As discussed in Chapter 20, exponential functions model situations in which a quantity is multiplied by a constant factor for each time period. An exponential function can be increasing with time, in which case it models exponential growth, or it can be decreasing with time, in which case it models exponential decay.

EXAMPLE 8

A researcher estimates that the population of a city is declining at an annual rate of 0.6%. If the current population of the city is 80,000, which of the following expressions appropriately models the population of the city t years from now according to the researcher's estimate?

A) $80{,}000(1 - 0.006)^t$
B) $80{,}000(1 - 0.006^t)$
C) $80{,}000 - 1.006^t$
D) $80{,}000(0.006^t)$

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A quantity that grows or decays by a fixed percent at regular intervals is said to possess exponential growth or decay.

Exponential growth is represented by the function $y = a(1 + r)^t$, while exponential decay is represented by the function $y = a(1 - r)^t$, where y is the new population, a is the initial population, r is the rate of growth or decay, and t is the number of time intervals that have elapsed.

According to the researcher's estimate, the population is decreasing by 0.6% each year. Since 0.6% is equal to 0.006, after the first year, the population is $80{,}000 - 0.006(80{,}000) = 80{,}000(1 - 0.006)$. After the second year, the population is $80{,}000(1 - 0.006) - 0.006(80{,}000)(1 - 0.006) = 80{,}000(1 - 0.006)^2$. Similarly, after t years, the population will be $80{,}000(1 - 0.006)^t$ according to the researcher's estimate. This is choice A.

Another well-known example of exponential decay is the decay of a radioactive isotope. One example is iodine-131, a radioactive isotope used in some medical treatments. The decay of iodine-131 emits beta and gamma radiation, and it decays to xenon-131. The half-life of iodine-131 is 8.02 days; that is, after 8.02 days, half of the iodine-131 in a sample will have decayed to xenon-131. Suppose a sample of A milligrams of iodine-131 decays for d days. Every 8.02 days, the quantity of iodine-131 is multiplied by $\frac{1}{2}$, or 2^{-1}. In d days, a total of $\frac{d}{8.02}$ different 8.02-day periods will have passed, and so the original quantity will have been multiplied by 2^{-1} a total of $\frac{d}{8.02}$ times. Therefore, the amount, in milligrams, of iodine-131 remaining in the sample will be $A(2^{-1})^{\frac{d}{8.02}} = A\left(2^{-\frac{d}{8.02}}\right)$.

In the preceding discussion, we used the identity $\frac{1}{2} - 2^{-1}$. Questions on the SAT Math Test may require you to apply this and other laws of exponents and the relationship between powers and radicals.

EXAMPLE 9

Which of the following is equivalent to $\left(\frac{1}{\sqrt{x}}\right)^n$?

A) $x^{\frac{n}{2}}$
B) $x^{-\frac{n}{2}}$
C) $x^{n+\frac{1}{2}}$
D) $x^{n-\frac{1}{2}}$

The square root $\sqrt{x}$ is equal to $x^{\frac{1}{2}}$. Thus, $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$, and $\left(\frac{1}{\sqrt{x}}\right)^n = \left(x^{-\frac{1}{2}}\right)^n = x^{-\frac{n}{2}}$. Choice B is the correct answer.

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Practice your exponent rules. Know, for instance, that $\sqrt{x} = x^{\frac{1}{2}}$ and that $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$.

An SAT Math Test question may also ask you to solve a radical equation. In solving radical equations, you may square both sides of an equation. Since squaring is *not* a reversible operation, you may end up with an extraneous root, that is, a root to the simplified equation that is *not* a root to the original equation. Thus, when solving a radical equation, you should check any solution you get in the original equation.

EXAMPLE 10

$$x - 12 = \sqrt{x + 44}$$

What is the solution set for the above equation?

A) {5}
B) {20}
C) {−5, 20}
D) {5, 20}

Squaring each side of $x - 12 = \sqrt{x + 44}$ gives

$$(x - 12)^2 = (\sqrt{x + 44})^2 = x + 44$$
$$x^2 + 24x + 144 = x + 44$$
$$x^2 - 25x + 100 = 0$$
$$(x - 5)(x - 20) = 0$$

The solutions to the quadratic are $x = 5$ and $x = 20$. However, since the first step was to square each side of the given equation, which is not a reversible operation, you need to check $x = 5$ and $x = 20$ in the original equation. Substituting 5 for x gives

$$5 - 12 = \sqrt{5 + 44}$$
$$-7 = \sqrt{49}$$

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A good strategy to use when solving radical equations is to square both sides of the equation. When doing so, however, be sure to check the solutions in the original equation, as you may end up with a root that is not a solution to the original equation.

This is not a true statement (since $\sqrt{49}$ represents only the positive square root, 7), so $x = 5$ is *not* a solution to $x - 12 = \sqrt{x + 44}$. Substituting 20 for x gives

$$20 - 12 = \sqrt{20 + 44}$$
$$8 = \sqrt{64}$$

This is a true statement, so $x = 20$ is a solution to $x - 12 = \sqrt{x + 44}$. Therefore, the solution set is {20}, which is choice B.

Dividing Polynomials by a Linear Expression and Solving Rational Equations

Questions on the SAT Math Test may assess your ability to work with rational expressions, including fractions with a variable in the denominator. This may include long division of a polynomial by a linear expression or finding the solution to a rational equation.

EXAMPLE 11

When $6x^2 - 5x + 4$ is divided by $3x + 2$, the result is $2x - 3 + \frac{R}{(3x + 2)}$, where R is a constant. What is the value of R?

Performing the long division gives

$$\begin{array}{r} 2x - 3 \\ 3x + 2 \overline{)6x^2 - 5x + 4} \\ \underline{6x^2 + 4x} \\ -9x + 4 \\ \underline{-9x - 6} \\ 10 \end{array}$$

Therefore, the remainder is 10.

If $ax + b$ is a factor of the polynomial $P(x)$, then $P(x)$ can be written as

$$P(x) = (ax + b)Q(x)$$

for some polynomial $Q(x)$. It follows that the solution to $ax + b = 0$, namely, $x = -\frac{b}{a}$, is a solution to $P(x) = 0$. More generally, if the number r is the remainder when $P(x)$ is divided by $ax + b$, you have

$$P(x) = (ax + b)Q(x) + r$$

It follows that for $x = -\frac{b}{a}$, the value of $P\left(-\frac{b}{a}\right) = (0)(Q(x)) + r = r$. This is another way to solve Example 11. The solution of $3x + 2 = 0$ is $x = -\frac{2}{3}$, so the remainder when $6x^2 - 5x + 4$ is divided by $3x + 2$ is the value of $6x^2 - 5x + 4$ when $-\frac{2}{3}$ is substituted for x: Remainder: $6\left(-\frac{2}{3}\right)^2 - 5\left(-\frac{2}{3}\right) + 4 = \frac{8}{3} + \frac{10}{3} + 4 = 10$.

EXAMPLE 12

$$\frac{3}{t+1} = \frac{2}{t+3} + \frac{1}{4}$$

If t is a solution to the equation above and $t > 0$, what is the value of t?

The first step in solving this equation is to clear the variable out of the denominators by multiplying each side by $(t + 1)(t + 3)$. This gives $3(t + 3) = 2(t + 1) + \frac{1}{4}(t + 1)(t + 3)$. Now multiply each side by 4 to get rid of the fraction: $12(t + 3) = 8(t + 1)(t + 3)$. Expanding all the products and moving all the terms to the right-hand side gives $0 = t^2 - 25$. Therefore, the solutions to the equation are $t = 5$ and $t = -5$. Since $t > 0$, the value of t is 5.

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When solving for a variable in an equation involving fractions, a good first step is to clear the variable out of the denominators of the fractions.

Systems of Equations

Questions on the SAT Math Test may ask you to solve a system of equations in two variables in which one equation is linear and the other equation is quadratic or another nonlinear equation.

EXAMPLE 13

$$3x + y = -3$$
$$(x + 1)^2 - 4(x + 1) - 6 = y$$

If (x, y) is a solution of the system of equations above and $y > 0$, what is the value of y?

The structure of the second equation suggests that $(x + 1)$ is a factor of the first equation. Subtracting $3x$ from each side of the first equation gives you $y = -3 - 3x$, which can be rewritten as $y = -3(x + 1)$. Substituting $-3(x + 1)$ for y in the second equation gives you $(x + 1)^2 - 4(x + 1) - 6 = -3(x + 1)$, which can be rewritten as $(x + 1)^2 - (x + 1) - 6 = 0$. The structure of this equation suggests that $x + 1$ can be treated as a variable. Factoring gives you $((x + 1) - 3)((x + 1) + 2) = 0$, or $(x - 2)(x + 3) = 0$. Thus, either $x = 2$, which gives $y = -3 - 3(2) = -9$; or $x = -3$, which gives $y = -3 - 3(-3) = -9$. Therefore, the solutions to the system are $(2, -9)$ and $(-3, 6)$. Since the question states that $y > 0$, the value of y is 6.

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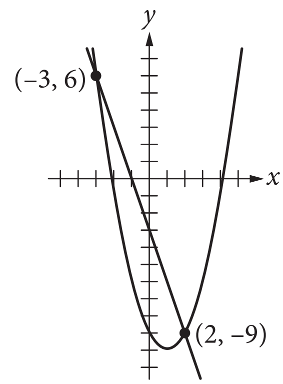
The first step to solving this example is substitution, an approach you may use on Heart of Algebra questions. The other key was noticing that $(x + 1)$ can be treated as a variable.



The solutions of the system are given by the intersection points of the two graphs. Questions on the SAT Math Test may assess this or other relationships between algebraic and graphical representations of functions.

Relationships Between Algebraic and Graphical Representations of Functions

A function $f(x)$ has a graph in the xy-plane, which is the graph of the equation $y = f(x)$ (or, equivalently, consists of all ordered pairs $(x, f(x))$). Some questions in Passport to Advanced Math assess your ability to relate properties of the function f to properties of its graph, and vice versa. You may be required to apply some of the following relationships:

- **Intercepts.** The x-intercepts of the graph of f correspond to values of x such that $f(x) = 0$; if the function f has no zeros, its graph has no x-intercepts, and vice versa. The y-intercept of the graph of f corresponds to the value of $f(0)$. If $x = 0$ is not in the domain of f, the graph of f has no y-intercept, and vice versa.

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The domain of a function is the set of all values for which the function is defined. The range of a function is the set of all values that are the output, or result, of applying the function.

- **Domain and range.** The domain of f is the set of all x for which $f(x)$ is defined. The range of f is the set of all y with $y = f(x)$ for some value of x in the domain. The domain and range can be found from the graph of f as the set of all x-coordinates and y-coordinates, respectively, of points on the graph.

- **Maximum and minimum values.** The maximum and minimum values of f can be found by locating the highest and the lowest points on the graph, respectively. For example, suppose P is the highest point on the graph of f. Then the y-coordinate of P is the maximum value of f, and the x-coordinate of P is where f takes on its maximum value.

- **Increasing and decreasing.** The graph of f shows the intervals over which the function f is increasing and decreasing.

- **End behavior.** The graph of f can indicate if $f(x)$ increases or decreases without limit as x gets very large and positive or very large and negative.

- **Asymptotes.** If the values of f approach a fixed value, say K, as x gets very large and positive or very large and negative, the graph of f has a horizontal asymptote at $y = K$. If f is a rational function whose denominator is zero and numerator is nonzero at $x = a$, then the graph of f has a vertical asymptote at $x = a$.

- **Symmetry.** If the graph of f is symmetric about the y-axis, then f is an even function, that is, $f(-x) = f(x)$ for all x in the domain of f. If the graph of f is symmetric about the origin, then f is an odd function, that is, $f(-x) = -f(x)$ for all x in the domain of f.

- **Transformations.** For a graph of a function f, a change of the form $f(x) + a$ will result in a vertical shift of a units and a change of the form $f(x + a)$ will result in a horizontal shift of a units.

Note: The SAT Math Test uses the following conventions about graphs in the xy-plane *unless* a particular question clearly states or shows a different convention:

- The axes are perpendicular.
- Scales on the axes are linear scales.
- The size of the units on the two axes *cannot* be assumed to be equal unless the question states they are equal or you are given enough information to conclude they are equal.
- The values on the horizontal axis increase as you move to the right.
- The values on the vertical axis increase as you move up.

REMEMBER

Don't assume the size of the units on the two axes are equal unless the question states they are equal or you can conclude they are equal from the information given.

EXAMPLE 14

The graph of which of the following functions in the xy-plane has x-intercepts at -4 and 5?

A) $f(x) = (x + 4)(x - 5)$
B) $g(x) = (x - 4)(x + 5)$
C) $h(x) = (x - 4)^2 + 5$
D) $k(x) = (x + 5)^2 - 4$

The x-intercepts of the graph of a function correspond to the zeros of the function. If a function has x-intercepts at -4 and 5, then the values of the function at -4 and 5 are each 0. The function in choice A is in factored form,

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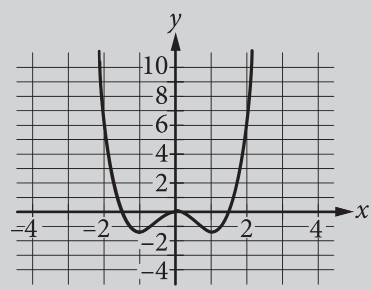
Another way to think of this question is to ask yourself, "Which answer choice represents a function that has values of zero when $x = -4$ and $x = +5$?"

which shows that $f(x) = 0$ if and only if $x + 4 = 0$ or $x - 5 = 0$, that is, if $x = -4$ or $x = 5$. Therefore, $f(x) = (x + 4)(x - 5)$ has x-intercepts at -4 and 5.

The graph in the xy-plane of each of the functions in the previous example is a parabola. Using the defining equations, you can tell that the graph of g has x-intercepts at 4 and -5; the graph of h has its vertex at (4, 5); and the graph of k has its vertex at $(-5, -4)$.

EXAMPLE 15



The function $f(x) = x^4 - 2.4x^2$ is graphed in the xy-plane as shown above. If k is a constant such that the equation $f(x) = k$ has 4 solutions, which of the following could be the value of k?

A) 1
B) 0
C) −1
D) −2

Choice C is correct. The equation $f(x) = k$ will have 4 solutions if and only if the graph of the horizontal line with equation $y = k$ intersects the graph of f at 4 points. The graph shows that of the given choices, only for choice C, -1, does the graph of $y = -1$ intersect the graph of f at 4 points.

Function Notation

The SAT Math Test assesses your understanding of function notation. You must be able to evaluate a function given the rule that defines it, and if the function describes a context, you may need to interpret the value of the function in the context. A question may ask you to interpret a function when an expression, such as $2x$ or $x + 1$, is used as the argument instead of the variable x, or a question may ask you to evaluate the composition of two functions.

EXAMPLE 16

If $g(x) = 2x + 1$ and $f(x) = g(x) + 4$, what is $f(2)$?

You are given $f(x) = g(x) + 4$ and therefore $f(2) = g(2) + 4$. To determine the value of $g(2)$, use the function $g(x) = 2x + 1$. Thus, $g(2) = 2(2) + 1$, and $g(2) = 5$. Substituting $g(2)$ gives $f(2) = 5 + 4$, or $f(2) = 9$.

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What may seem at first to be a complex question boils down to straightforward substitution.

Analyzing More Complex Equations in Context

Equations and functions that describe a real-life context can be complex. Often it is not possible to analyze them as completely as you can analyze a linear equation or function. You still can acquire key information about the context by analyzing the equation or function that describes it. Questions on the Passport to Advanced Math section may ask you to use an equation describing a context to determine how a change in one quantity affects another quantity. You may also be asked to manipulate an equation to isolate a quantity of interest on one side of the equation. You may be asked to produce or identify a form of an equation that reveals new information about the context it represents or about the graphical representation of the equation.

EXAMPLE 17

If an object of mass m is moving at speed v, the object's kinetic energy KE is given by the equation $KE = \frac{1}{2}mv^2$. If the mass of the object is halved and its speed is doubled, how does the kinetic energy change?

A) The kinetic energy is halved.
B) The kinetic energy is unchanged.
C) The kinetic energy is doubled.
D) The kinetic energy is quadrupled (multiplied by a factor of 4).

Choice C is correct. If the mass of the object is halved, the new mass is $\frac{m}{2}$. If the speed of the object is doubled, its new speed is $2v$. Therefore, the new kinetic energy is $\frac{1}{2}\left(\frac{m}{2}\right)(2v)^2 = \frac{1}{2}\left(\frac{m}{2}\right)(4v^2) = mv^2$. This is double the kinetic energy of the original object, which was $\frac{1}{2}mv^2$.

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Another way to check your answer here is to pick simple numbers for mass and speed and examine the impact on kinetic energy when those values are altered as indicated by the question. If mass and speed both equal 1, kinetic energy is $\frac{1}{2}$. When mass is halved, to $\frac{1}{2}$, and speed is doubled, to 2, the new kinetic energy is 1. Since 1 is twice the value of $\frac{1}{2}$, we know that kinetic energy is doubled.

EXAMPLE 18

A gas in a container will escape through holes of microscopic size, as long as the holes are larger than the gas molecules. This process is called effusion. If a gas of molar mass M_1 effuses at a rate of r_1 and a gas of molar mass M_2 effuses at a rate of r_2, then the following relationship holds.

$$\frac{r_1}{r_2} = \sqrt{\frac{M_2}{M_1}}$$

This is known as Graham's law. Which of the following correctly expresses M_2 in terms of M_1, r_1, and r_2?

A) $M_2 = M_1 \frac{r_1^2}{r_2^2}$

B) $M_2 = M_1 \frac{r_2^2}{r_1^2}$

C) $M_2 = \sqrt{M_1} \frac{r_1}{r_2}$

D) $M_2 = \sqrt{M_1} \frac{r_2}{r_1}$

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Always start by identifying exactly what the question asks. In this case, you are being asked to isolate the variable M_2. Squaring both sides of the equation is a great first step as it allows you to eliminate the radical sign.

Squaring each side of $\frac{r_1}{r_2} = \sqrt{\frac{M_2}{M_1}}$ gives $\left(\frac{r_1}{r_2}\right)^2 = \left(\sqrt{\frac{M_2}{M_1}}\right)^2$, which can be rewritten as $\frac{M_2}{M_1} = \frac{r_1^2}{r_2^2}$. Multiplying each side of $\frac{M_2}{M_1} = \frac{r_1^2}{r_2^2}$ by M_1 gives $M_2 = M_1 \frac{r_1^2}{r_2^2}$, which is choice A.

EXAMPLE 19

A store manager estimates that if a video game is sold at a price of p dollars, the store will have weekly revenue, in dollars, of $r(p) = -4p^2 + 200p$ from the sale of the video game. Which of the following equivalent forms of $r(p)$ shows, as constants or coefficients, the maximum possible weekly revenue and the price that results in the maximum revenue?

A) $r(p) = 200p - 4p^2$
B) $r(p) = -2(2p^2 - 100p)$
C) $r(p) = -4(p^2 - 50p)$
D) $r(p) = -4(p - 25)^2 + 2{,}500$

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The fact that the coefficient of the squared term is negative for this function indicates that the graph of r in the coordinate plane is a parabola that opens downward. Thus, the maximum value of revenue corresponds to the vertex of the parabola.

Choice D is correct. The graph of r in the coordinate plane is a parabola that opens downward. The maximum value of revenue corresponds to the vertex of the parabola. Since the square of any real number is always nonnegative, the form $r(p) = -4(p - 25)^2 + 2{,}500$ shows that the vertex of the parabola is (25, 2,500); that is, the maximum must occur where $-4(p - 25)^2$ is 0, which is $p = 25$, and this maximum is $r(25) = 2{,}500$. Thus, the maximum possible weekly revenue and the price that results in the maximum revenue occur as constants in the form $r(p) = -4(p - 25)^2 + 2{,}500$.